# The Halting Problem

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# Entscheidungsproblem revisited again

If a problem can be formally defined, is it always true that every instance of the problem be decided by mechanical reasoning?

Formal definition = Mathematical logic

Mechanical reasoning = TM

We will outline the proof Alan Turing gave to show the negative answer to Entscheidungsproblem.

# The accepting problem

$$\text{accept}(M, x) = \begin{cases} 1 & \text{if M accepts input x} \\ 0 & \text{else} \end{cases}$$

where:

M is a TM for a decision problem

x is an input string.

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# The undecidability of accept(M, x)

**Theorem**

accept(M, x) cannot be implemented by any TM.

What is the difference between accept(M, x) and SIM(M, x)?

SIM(M, x) can be done by the UTM.

- SIM(M, x) may not halt
- But accept(M, x) by definition will always halt.

# Proof (by contradiction)

Assume that $M^{accept}$ solves the accept problem.

Then we can construct the string ENC($M^{accept}$)

**Define:**

D(M) = ! accept(M, ENC(M))

Since accept is decidable, D is also decidable.

$M^D$ (M) = ! $M^{accept}$(M, ENC(M))

# Proof (by contradiction)

Challenge: $D(M^D) = ?$

Does $D(M^D) = 0$ ?

| | | |
|---|---|---|
| $\Rightarrow$ | $accept(M^D, ENC(M^D)) = 1$ | by definition of D |
| $\Rightarrow$ | $M^D$ accepts $END(M^D)$ | by definition of accept |
| $\Rightarrow$ | $M^D(ENC(M^D)) = 1$ | |
| $\Rightarrow$ | $D(M^D) = 1$ | since $M^D$ implements D |
| $\Rightarrow$ | *contradiction* | |

# Proof (by contradiction)

Challenge: $D(M^D) = ?$

Does $D(M^D) = 1$ ?

| | | |
|---|---|---|
| $\Rightarrow$ | $accept(M^D, ENC(M^D)) = 0$ | by definition of D |
| $\Rightarrow$ | $M^D$ does not accepts $END(M^D)$ | by definition of accept |
| $\Rightarrow$ | $M^D(ENC(M^D)) = 0$ | |
| $\Rightarrow$ | $D(M^D) = 0$ | since $M^D$ implements D |
| $\Rightarrow$ | *contradiction* | |

# Proof (by contradiction)

*The definition of accept always guarantees Maccept halts. But for SIM, there is another possibility:*

*SIM(M, ENC(M)) does not halt.*

Challenge: $D(M^D) = ?$

Does $D(M^D) = 0$ ?

$\Rightarrow$ $accept(M^D, ENC(M^D)) = 1$

$\Rightarrow$ $M^D$ accepts $END(M^D)$

$\Rightarrow$ $M^D(ENC(M^D)) = 1$

$\Rightarrow$ $D(M^D) = 1$

$\Rightarrow$ *contradiction*

Does $D(M^D) = 1$ ?

$\Rightarrow$ $accept(M^D, ENC(M^D)) = 0$

$\Rightarrow$ $M^D$ does not accepts $END(M^D)$

$\Rightarrow$ $M^D(ENC(M^D)) = 0$

$\Rightarrow$ $D(M^D) = 0$

$\Rightarrow$ *contradiction*

# Contradiction in all cases:

So, the hypothesis that $M^{accept}$ exists cannot be true.

Thus, the ACCEPT problem is undecidable.

# The Halting problem

$$\text{halt}(M, x) = \begin{cases} 1 & \text{if M will halt} \\ 0 & \text{else} \end{cases}$$

**Theorem**:

The halting problem is undecidable.

**Proof**:

Observe:

```
accept(M, x) =
    if halt(M, x) then
        SIM(M, x)
    else
        0
    end if
```

So, we can use $M^{halt}$ to construct $M^{accept}$.